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Drinfeld twists and algebraic Bethe ansatz of the supersymmetric t-J model

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ABSTRACT: We construct the Drinfeld twists (factorizing F-matrices) for the supersymmetric t-J model. Working in the basis provided by the F-matrix (i.e. the so-called F-basis), we obtain completely symmetric representations of the monodromy matrix and the pseudo-particle creation operators of the model. These enable us to resolve the hierarchy of the nested Bethe vectors for the $gl(2|1)$ invariant t-J model.

KEYWORDS: Lattice Integrable Models, Bethe Ansatz.

Contents

1. Introduction

The algebraic Bethe ansatz or the quantum inverse scattering method (QISM) provides a powerful tool of solving eigenvalue problems such as diagonalizing integrable two-dimensional quantum spin chains. In this framework, the pseudo-particle creation and annihilation operators are constructed by the off-diagonal entries of the monodromy matrix. The Bethe vectors (eigenvectors) are obtained by acting the creation operators on the pseudo-vacuum state. However, the apparently simple action of creation operators is intricate on the level of the local operators by non-local effects arising from polalization clouds or compensating exchange terms. This makes the exact and explicit computation of correlation functions difficult (if not impossible).

Recently, Maillet and Sanchez de Santos [1] showed how monodromy matrices of the inhomogeneous XXX and XXZ spin chains can be simplified by using the factorizing Drinfeld twists. This leads to the natural F-basis for the analysis of these models. In this basis, the pseudo-particle creation and annihilation operators take completely symmetric forms and contain no compensating exchange terms on the level of the local operators (i.e. polarization free). As a result, the Bethe vectors of the models are simplified dramatically and can be written down explicitly.

The results of [1] were generalized to certain other systems. In [3], the Drinfeld twists associated with any finite-dimensional irreducible representations of the yangian $Y[gl(2)]$ were investigated. In [4], the form factors for local spin operators of the spin-1/2 XXZ model were computed and in [5], the spontaneous magnetization of the XXZ chain on the finite lattice was represented. In [2], Albert et al constructed the F-matrix of the $gl(m)$ rational Heisenberg model and obtained a polarization free representation of the creation operators. Using these results, they resolved the hierarchy of the nested Bethe ansatz for the $gl(m)$ model. In [6, 7], the Drinfeld twists of the elliptic XYZ model and Belavin model were constructed.

The t-J model was proposed in an attempt to understand high-T_c superconductivity [8, 9, 10]. It is a strongly correlated electron system with nearest-neighbor hopping (t) and anti-ferromagnetic exchange (J) of electrons. When $J = 2t$, the t-J model becomes $gl(2|1)$ invariant. Using the nested algebraic Bethe ansatz method, Essler and Korepin obtained the eigenvalues of the supersymmetric t-J model [11]. The algebraic structure of the model was investigated in [12].

In this paper, we construct the factorizing F-matrix of the supersymmetric t-J model. Working in the F-basis, we obtain the symmetric representations of the monodromy matrix and the creation operators. Using these results, we resolve the hierarchy of the nested Bethe vectors of the $gl(2|1)$ invariant t-J model.

The present paper is organized as follows. In section 2, we introduce some basic notation of the supersymmetric t-J model. In section 3, we construct the F-matrix and its inverse. In section 4, we give the representation of the monodromy matrix and the creation operators in the F-basis. The nested Bethe vectors of the model are resolved in section 5. We conclude the paper by offering some discussions in section 6.

2. Basic definitions and notation

Let V be the 3-dimensional $gl(2|1)$-module and $R \in End(V \otimes V)$ the R-matrix associated with this module. V is Z_2-graded, and in the following we choose the BBF grading for V, i.e. $[1] = [2] = 0, [3] = 1$. The R-matrix depends on the difference of two spectral parameters u_1 and u_2 associated with the two copies of V, and is, in the BBF grading, given by

$$R_{12}(u_1, u_2) = R_{12}(u_1 - u_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & 0 & 0 & 0 & b_{12} & 0 & 0 \\ 0 & b_{12} & 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{12} & 0 & b_{12} & 0 \\ 0 & 0 & b_{12} & 0 & 0 & 0 & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{12} & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{12} \end{pmatrix}, \qquad (2.1)$$

where

$$
\begin{aligned}
a_{12} &= a(u_1, u_2) \equiv \frac{u_1 - u_2}{u_1 - u_2 + \eta}\,, \qquad b_{12} = b(u_1, u_2) \equiv \frac{\eta}{u_1 - u_2 + \eta}\,, \\
c_{12} &= c(u_1, u_2) \equiv \frac{u_1 - u_2 - \eta}{u_1 - u_2 + \eta}
\end{aligned} \tag{2.2}
$$

with $\eta \in C$ being the crossing parameter. One can easily check that the R-matrix satisfies the unitary relation

$$
R_{21} R_{12} = 1\,. \tag{2.3}
$$

Here and throughout $R_{12} \equiv R_{12}(u_1, u_2)$. The R-matrix satisfies the graded Yang-Baxter equation (GYBE)

$$
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}\,. \tag{2.4}
$$

In terms of the matrix elements defined by

$$
R(u)(v^{i'} \otimes v^{j'}) = \sum_{i,j} R(u)_{ij}^{i'j'} (v^i \otimes v^j)\,, \tag{2.5}
$$

the GYBE reads

$$
\begin{aligned}
&\sum_{i',j',k'} R(u_1 - u_2)_{ij}^{i'j'} R(u_1 - u_3)_{i'k}^{i''k'} R(u_2 - u_3)_{j'k'}^{j''k''} (-1)^{[j']([i']+[i''])} = \\
&\quad = \sum_{i',j',k'} R(u_2 - u_3)_{jk}^{j'k'} R(u_1 - u_3)_{ik'}^{i'k''} R(u_1 - u_2)_{i'j'}^{i''j''} (-1)^{[j']([i]+[i'])}\,.
\end{aligned} \tag{2.6}
$$

The quantum monodromy matrix $T(u)$ of the supersymmetric t-J chain of length N is defined as

$$
T(u) = R_{0N}(u, z_N) R_{0N-1}(u, z_{N-1}) ... R_{01}(u, z_1)\,, \tag{2.7}
$$

where the index 0 refers to the auxiliary space and $\{z_i\}$ are arbitrary inhomogeneous parameters depending on site i. $T(u)$ can be represented in the auxiliary space as the 3×3 matrix whose elements are operators acting on the quantum space $V^{\otimes N}$:

$$
T(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) & B_1(u) \\ A_{21}(u) & A_{22}(u) & B_2(u) \\ C_1(u) & C_2(u) & D(u) \end{pmatrix}. \tag{2.8}
$$

By using the GYBE, one may prove that the monodromy matrix satisfies the GYBE

$$
R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v)\,. \tag{2.9}
$$

Define the transfer matrix $t(u)$

$$
t(u) = str_0 T(u)\,, \tag{2.10}
$$

where str_0 denotes the supertrace over the auxiliary space. Then the hamiltonian of the supersymmetric t-J model is given by

$$
H = \frac{d \ln t(u)}{du}|_{u=0}\,. \tag{2.11}
$$

This model is integrable thanks to the commutativity of the transfer matrix for different parameters,

$$[t(u), t(v)] = 0 \,, \tag{2.12}$$

which can be verified by using the GYBE.

Following [1], we now introduce the notation $R^{\sigma}_{1\ldots N}$, where σ is any element of the permutation group $\mathcal{S}_N$. We note that we may rewrite the GYBE as

$$R^{\sigma_{23}}_{23} T_{0,23} = T_{0,32} R^{\sigma_{23}}_{23} \,, \tag{2.13}$$

where $T_{0,23} \equiv R_{03}R_{02}$ and σ_{23} is the transposition of space labels (2,3). It follows that $R^{\sigma}_{1\ldots N}$ is a product of elementary R-matrices, corresponding to a decomposition of σ into elementary transpositions. With the help of the GYBE, one may generalize (2.13) to a N-fold tensor product of spaces

$$R^{\sigma}_{1\ldots N} T_{0,1\ldots N} = T_{0,\sigma(1\ldots N)} R^{\sigma}_{1\ldots N} \,, \tag{2.14}$$

where $T_{0,1\ldots N} \equiv R_{0N} \ldots R_{01}$. This implies the "decomposition" law

$$R^{\sigma'\sigma}_{1\ldots N} = R^{\sigma}_{\sigma'(1\ldots N)} R^{\sigma'}_{1\ldots N} \,, \tag{2.15}$$

for a product of two elements in $\mathcal{S}_N$. Note that $R^{\sigma}_{\sigma'(1\ldots N)}$ satisfies the relation

$$R^{\sigma}_{\sigma'(1\ldots N)} T_{0,\sigma'(1\ldots N)} = T_{0,\sigma'\sigma(1\ldots N)} R^{\sigma}_{\sigma'(1\ldots N)} \,. \tag{2.16}$$

As in [2], we write the elements of $R^{\sigma}_{1\ldots N}$ as

$$(R^{\sigma}_{1\ldots N})^{\alpha_{\sigma(N)}\ldots\alpha_{\sigma(1)}}_{\beta_N\ldots\beta_1} \,, \tag{2.17}$$

where the labels in the upper indices are permuted relative to the lower indices according to σ.

3. F-matrices for the supersymmetric t-J model

In [1], Maillet and Sanchez de Santos constructed the Drinfeld factorizing twists, i.e. the so-called factorizing F-matrices, of the XXX model:

$$R_{12} = F^{-1}_{21} F_{12} \,. \tag{3.1}$$

In [2], Albert et al generalized the results in [1] to the $gl(m)$ spin chain system. In this section, we construct the F-matrices associated with the supersymmetric t-J model.

3.1 The F-matrix

For the R-matrix (2.1), we define the F-matrix

$$F_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{12} & 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_{12} & 0 & 0 & 0 & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{12} & 0 & a_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+c_{12} \end{pmatrix}. \tag{3.2}$$

It is convenient to write the F-matrix as the form

$$F_{12} = \sum_{3\geq\alpha_2\geq\alpha_1} P_1^{\alpha_1} P_2^{\alpha_2} + c_{12} P_1^3 P_2^3 + \sum_{3\geq\alpha_1\rangle\alpha_2} P_1^{\alpha_1} P_2^{\alpha_2} R_{12}\,, \tag{3.3}$$

where $(P_i^{\alpha})_k^l = \delta_{k,\alpha}\delta_{l,\alpha}$ is the projector acting on ith space. Then by the R-matrix (2.1) and F-matrix (3.3), we have

$$\begin{aligned}
F_{21}R_{12} &= \left(\sum_{3\geq\alpha_1\geq\alpha_2} P_2^{\alpha_2} P_1^{\alpha_1} + c_{21} P_2^3 P_1^3 + \sum_{3\geq\alpha_2\rangle\alpha_1} P_2^{\alpha_2} P_1^{\alpha_1} R_{21}\right) R_{12} \\
&= \left(\sum_{3\geq\alpha_1\rangle\alpha_2} P_2^{\alpha_2} P_1^{\alpha_1} + P_2^1 P_1^1 + P_2^2 P_1^2 + (1+c_{21}) P_2^3 P_1^3 + \sum_{3\geq\alpha_2\rangle\alpha_1} P_2^{\alpha_2} P_1^{\alpha_1} R_{21}\right) R_{12} \\
&= \sum_{3\geq\alpha_1\rangle\alpha_2} P_2^{\alpha_2} P_1^{\alpha_1} R_{12} + P_2^1 P_1^1 + P_2^2 P_1^2 + (1+c_{12}) P_2^3 P_1^3 + \sum_{3\geq\alpha_2\rangle\alpha_1} P_2^{\alpha_2} P_1^{\alpha_1} \\
&= \sum_{3\geq\alpha_1\rangle\alpha_2} P_2^{\alpha_2} P_1^{\alpha_1} R_{12} + c_{12} P_2^3 P_1^3 + \sum_{3\geq\alpha_2\geq\alpha_1} P_2^{\alpha_2} P_1^{\alpha_1} \\
&= F_{12}.
\end{aligned} \tag{3.4}$$

Here we have used $R_{12}R_{21} = 1$ and $c_{12}c_{21} = 1$. Some remarks are in order. The solutions to (3.1), i.e. the F-matrices satisfying (3.1), are not unique [1, 2]. In this paper, we only consider the particular solution (3.3), which is lower-triangle.

We now generalize the F-matrix to the N-site problem. As is pointed out in [2], the generalized F-matrix should satisfy the three properties: i) lower-triangularity; ii) non-degeneracy and

$$\text{iii)}\quad F_{\sigma(1\ldots N)}(u_{\sigma(1)},\ldots,u_{\sigma(N)})R^{\sigma}_{1\ldots N}(u_1,\ldots,u_N) = F_{1\ldots N}(u_1,\ldots,u_N)\,, \tag{3.5}$$

where $\sigma \in \mathcal{S}_N$ and u_i, $i = 1,\ldots,N$, are generic inhomogeneous parameters .

Define the N-site F-matrix:

$$F_{1,\ldots N} = \sum_{\sigma\in\mathcal{S}_N} \sum^{*}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(N)}} \prod_{j=1}^{N} P_{\sigma(j)}^{\alpha_{\sigma(j)}} S(c,\sigma,\alpha_\sigma) R^{\sigma}_{1\ldots N}\,, \tag{3.6}$$

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where the sum $\sum^*$ is over all non-decreasing sequences of the labels $\alpha_{\sigma(i)}$:

$$\begin{aligned}
&\alpha_{\sigma(i+1)} \geq \alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1)\rangle\sigma(i) \\
&\alpha_{\sigma(i+1)}\rangle\alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1) < \sigma(i)
\end{aligned} \tag{3.7}$$

and the c-number function $S(c, \sigma, \alpha_\sigma)$ is given by

$$S(c, \sigma, \alpha_\sigma) \equiv \exp\left\{ \sum_{l>k=1}^{N} \delta^{[3]}_{\alpha_{\sigma(k)}, \alpha_{\sigma(l)}} \ln(1 + c_{\sigma(k)\sigma(l)}) \right\} \tag{3.8}$$

with $\delta^{[3]}_{\alpha_{\sigma(k)},\alpha_{\sigma(l)}} = 1$ for $\alpha_{\sigma(k)} = \alpha_{\sigma(l)} = 3$, and $\delta^{[3]}_{\alpha_{\sigma(k)},\alpha_{\sigma(l)}} = 0$ otherwise.

The definition of $F_{1\ldots N}$, (3.6), and the summation condition (3.7) imply that $F_{1\ldots N}$ is a lower-triangular matrix. Moreover, one can easily check that the F-matrix is non-degenerate because all diagonal elements are non-zero.

We now prove that the F-matrix (3.6) satisfies the property iii). Any given permutation $\sigma \in \mathcal{S}_N$ can be decomposed into elementary transpositions of the group $\mathcal{S}_N$ as $\sigma = \sigma_1 \ldots \sigma_k$ with σ_i denoting the elementary permutation $(i, i+1)$. By (2.15), we have if the property iii) holds for elementary transposition σ_i,

$$\begin{aligned}
F_{\sigma(1\ldots N)} R^{\sigma}_{1\ldots N} &= \\
&= F_{\sigma_1\ldots\sigma_k(1\ldots N)} R^{\sigma_k}_{\sigma_1\ldots\sigma_{k-1}(1\ldots N)} R^{\sigma_{k-1}}_{\sigma_1\ldots\sigma_{k-2}(1\ldots N)} \cdots R^{\sigma_1}_{1\ldots N} \\
&= F_{\sigma_1\ldots\sigma_{k-1}(1\ldots N)} R^{\sigma_{k-1}}_{\sigma_1\ldots\sigma_{k-2}(1\ldots N)} \cdots R^{\sigma_1}_{1\ldots N} \\
&= \cdots = F_{\sigma_1(1\ldots N)} R^{\sigma_1}_{1\ldots N} = F_{1\ldots N}.
\end{aligned} \tag{3.9}$$

For the elementary transposition σ_i, we have

$$\begin{aligned}
F_{\sigma_i(1\ldots N)} R^{\sigma_i}_{1\ldots N} &= \\
&= \sum_{\sigma \in \mathcal{S}_N} \sum^{*}_{\alpha_{\sigma_i\sigma(1)}\ldots\alpha_{\sigma_i\sigma(N)}} \prod_{j=1}^{N} P^{\alpha_{\sigma_i\sigma(j)}}_{\sigma_i\sigma(j)} S(c, \sigma_i\sigma, \alpha_{\sigma_i\sigma}) R^{\sigma}_{\sigma_i(1\ldots N)} R^{\sigma_i}_{1\ldots N} \\
&= \sum_{\sigma \in \mathcal{S}_N} \sum^{*}_{\alpha_{\sigma_i\sigma(1)}\ldots\alpha_{\sigma_i\sigma(N)}} \prod_{j=1}^{N} P^{\alpha_{\sigma_i\sigma(j)}}_{\sigma_i\sigma(j)} S(c, \sigma_i\sigma, \alpha_{\sigma_i\sigma}) R^{\sigma_i\sigma}_{1\ldots N} \\
&= \sum_{\tilde\sigma \in \mathcal{S}_N} \sum^{*(i)}_{\alpha_{\tilde\sigma(1)}\ldots\alpha_{\tilde\sigma(N)}} \prod_{j=1}^{N} P^{\alpha_{\tilde\sigma(j)}}_{\tilde\sigma(j)} S(c, \tilde\sigma, \alpha_{\tilde\sigma}) R^{\tilde\sigma}_{1\ldots N}\,,
\end{aligned} \tag{3.10}$$

where $\tilde\sigma = \sigma_i\sigma$, and the summation sequences of $\alpha_{\tilde\sigma}$ in $\sum^{*(i)}$ now has the form

$$\begin{aligned}
&\alpha_{\tilde\sigma(j+1)} \geq \alpha_{\tilde\sigma(j)} \quad \text{if} \quad \sigma_i\tilde\sigma(j+1) > \sigma_i\tilde\sigma(j)\,, \\
&\alpha_{\tilde\sigma(j+1)} > \alpha_{\tilde\sigma(j)} \quad \text{if} \quad \sigma_i\tilde\sigma(j+1) < \sigma_i\tilde\sigma(j)\,.
\end{aligned} \tag{3.11}$$

Comparing (3.11) with (3.7), we find that the only difference between them is the transposition σ_i factor in the "if" conditions. For a given $\tilde\sigma \in \mathcal{S}_N$ with $\tilde\sigma(j) = i$ and $\tilde\sigma(k) = i+1$, we now examine how the elementary transposition σ_i will affect the inequalities (3.11).

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1) If $|j-k|>1$, then σ_i does not affect the sequence of $\alpha_{\tilde\sigma}$ at all, that is, the sign of inequality " $>$ " or "$\geq$" between two neighboring root indexes is unchanged with the action of σ_i.

2) If $|j-k|=1$, then in the summation sequences of $\alpha_{\tilde\sigma}$, when $\tilde\sigma(j+1)=i+1$ and $\tilde\sigma(j)=i$, sign "$\geq$" changes to " $>$ ", while when $\tilde\sigma(j+1)=i$ and $\tilde\sigma(j)=i+1$, " $>$ " changes to "$\geq$". Thus (3.7) and (3.10) differ only when equal labels $\alpha_{\tilde\sigma}$ appear. In the following, we study this difference between (3.6) and (3.10). Rewriting $\tilde\sigma$ as σ, and then subtracting $F_{1\ldots N}$ from $F_{\sigma_i(1\ldots N)}R^{\sigma_i}_{1\ldots N}$, we have

$$
\begin{aligned}
&F_{\sigma_i(1\ldots N)}R^{\sigma_i}_{1\ldots N}-F_{1\ldots N}=\\
&=\sum_{\substack{\sigma(j)=i+1\\ \sigma(j+1)=i}}\ \sum_{\alpha_{\sigma(j)}=\alpha_{\sigma(j+1)}=1}^{3}\ \sum^{*}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(j-1)}\alpha_{\sigma(j+2)}\cdots\alpha_{\sigma(N)}} P^{\alpha_{\sigma(1)}}_{\sigma(1)}\cdots P^{\alpha_{i+1}}_{i+1}P^{\alpha_i}_{i}\cdots P^{\alpha_{\sigma(N)}}_{\sigma(N)}\times\\
&\quad\times S(c,\sigma,\alpha_\sigma)R^{\sigma}_{1\ldots N}-\\
&\quad-\sum_{\substack{\sigma(j)=i\\ \sigma(j+1)=i+1}}\ \sum_{\alpha_{\sigma(j)}=\alpha_{\sigma(j+1)}=1}^{3}\ \sum^{*}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(j-1)}\alpha_{\sigma(j+2)}\cdots\alpha_{\sigma(N)}} P^{\alpha_{\sigma(1)}}_{\sigma(1)}\cdots\\
&\qquad\qquad\cdots P^{\alpha_i}_{i}P^{\alpha_{i+1}}_{i+1}\cdots P^{\alpha_{\sigma(N)}}_{\sigma(N)}\times\\
&\quad\times S(c,\sigma,\alpha_\sigma)R^{\sigma}_{1\ldots N} \qquad (3.12)\\
&=\sum_{\substack{\sigma(j)=i+1\\ \sigma(j+1)=i}}\ \sum_{\alpha_{i+1}=\alpha_i=1}^{3}\ \sum^{*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(j-1)}\alpha_{\sigma(j+2)}\ldots\alpha_{\sigma(N)}} P^{\alpha_{\sigma(1)}}_{\sigma(1)}\ldots P^{\alpha_{i+1}}_{i+1}P^{\alpha_i}_{i}\ldots P^{\alpha_{\sigma(N)}}_{\sigma(N)}\times\\
&\quad\times\exp\left\{\sum_{\substack{1=k<l\\ k,l\neq j,j+1}}^{N}\delta^{[3]}_{\alpha_{\sigma(k)},\alpha_{\sigma(l)}}\ln(1+c_{\sigma(k),\sigma(l)})\right\}\exp\left\{\delta^{[3]}_{\alpha_{i+1},\alpha_i}\ln(1+c_{i+1,i})\right\}R^{\sigma}_{1\ldots N}-\\
&\quad-\sum_{\substack{\sigma(j)=i\\ \sigma(j+1)=i+1}}\ \sum_{\alpha_i=\alpha_{i+1}=1}^{3}\ \sum^{*}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(j-1)}\alpha_{\sigma(j+2)}\cdots\alpha_{\sigma(N)}} P^{\alpha_{\sigma(1)}}_{\sigma(1)}\cdots P^{\alpha_i}_{i}P^{\alpha_{i+1}}_{i+1}\cdots P^{\alpha_{\sigma(N)}}_{\sigma(N)}\times\\
&\quad\times\exp\left\{\sum_{\substack{1=k<l\\ k,l\neq j,j+1}}^{N}\delta^{[3]}_{\alpha_{\sigma(k)},\alpha_{\sigma(l)}}\ln(1+c_{\sigma(k),\sigma(l)})\right\}\times\\
&\quad\times\exp\left\{\delta^{[3]}_{\alpha_i,\alpha_{i+1}}\ln(1+c_{i,i+1})\right\}R^{\sigma}_{1\ldots N}\,. \qquad (3.13)
\end{aligned}
$$

Making the change $\sigma\to\sigma\sigma_j$ in the first term of the r.h.s. and using (2.15), we have

$$
\begin{aligned}
&F_{\sigma_i(1\ldots N)}R^{\sigma_i}_{1\ldots N}-F_{1\ldots N}=\\
&=\sum_{\substack{\sigma(j)=i\\ \sigma(j+1)=i+1}}\ \sum^{*}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(j-1)}\alpha_{\sigma(j+2)}\cdots\alpha_{\sigma(N)}} P^{\alpha_{\sigma(1)}}_{\sigma(1)}\ldots P^{\alpha_{\sigma(i-1)}}_{\sigma(i-1)}P^{\alpha_{\sigma(i+2)}}_{\sigma(i+2)}\ldots P^{\alpha_{\sigma(N)}}_{\sigma(N)}\times
\end{aligned}
$$

$$\times \exp\left\{\sum_{\substack{1=k<l\\k,l\neq j,j+1}}^{N}\delta^{[3]}_{\alpha_{\sigma(k)},\alpha_{\sigma(l)}}\ln(1+c_{\sigma(k),\sigma(l)})\right\}\times$$
$$\times\left[\sum_{\alpha_{i+1}=\alpha_i=1}^{3}\left(P^{\alpha_{i+1}}_{i+1}P^{\alpha_i}_{i}\exp\left\{\delta^{[3]}_{\alpha_{i+1},\alpha_i}\ln(1+c_{i+1,i})\right\}R^{\sigma_j}_{\sigma(1\ldots N)}-\right.\right.$$
$$\left.\left.-P^{\alpha_i}_{i}P^{\alpha_{i+1}}_{i+1}\exp\left\{\delta^{[3]}_{\alpha_i,\alpha_{i+1}}\ln(1+c_{i,i+1})\right\}\right)\right]R^{\sigma}_{1\ldots N}\,. \tag{3.14}$$

Denoted by X the quantity in the square bracket. Then we have

$$\begin{aligned} X &= \left(P^1_{i+1}P^1_i+P^2_{i+1}P^2_i+(1+c_{i+1,i})P^3_{i+1}P^3_i\right)R^{\sigma_j}_{\sigma(1\ldots N)}-\\ &\quad-\left(P^1_iP^1_{i+1}+P^2_iP^2_{i+1}+(1+c_{i,i+1})P^3_iP^3_{i+1}\right)\\ &= \left(P^1_{i+1}P^1_i+P^2_{i+1}P^2_i+(1+c_{i+1,i})P^3_{i+1}P^3_i\right)R_{i,i+1}-\\ &\quad-\left(P^1_iP^1_{i+1}+P^2_iP^2_{i+1}+(1+c_{i,i+1})P^3_iP^3_{i+1}\right)\\ &= 0\,. \end{aligned} \tag{3.15}$$

Thus, we obtain

$$R^{\sigma}_{1\ldots N}(u_1,\ldots,u_N)=F^{-1}_{\sigma(1\ldots N)}(u_{\sigma(1)},\ldots,u_{\sigma(N)})F_{1\ldots N}(u_1,\ldots,u_N)\,. \tag{3.16}$$

The factorizing F-matrix $F_{1\ldots N}$ of the supersymmetric t-J model is proved to satisfy all three properties.

3.2 The inverse $F^{-1}_{1\ldots N}$ of the F-matrix

The non-degenerate property of the F-matrix implies that we can find the inverse matrix $F^{-1}_{1\ldots N}$. To do so, we first define the F^*-matrix

$$F^*_{1\ldots N}=\sum_{\sigma\in\mathcal{S}_N}\sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}^{**}S(c,\sigma,\alpha_\sigma)R^{\sigma^{-1}}_{\sigma(1\ldots N)}\prod_{j=1}^{N}P^{\alpha_{\sigma(j)}}_{\sigma(j)}, \tag{3.17}$$

where the sum $\sum^{**}$ is taken over all possible α_i which satisfies the following non-increasing constraints:

$$\begin{aligned} &\alpha_{\sigma(i+1)}\leq\alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1)<\sigma(i)\,,\\ &\alpha_{\sigma(i+1)}<\alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1)>\sigma(i)\,. \end{aligned} \tag{3.18}$$

Now we compute the product of $F_{1\ldots N}$ and $F^*_{1\ldots N}$. Substituting (3.6) and (3.17) into the product, we have

$$F_{1\ldots N}F^*_{1\ldots N}=\sum_{\sigma\in\mathcal{S}_N}\sum_{\sigma'\in\mathcal{S}_N}\sum_{\alpha_{\sigma_1}\ldots\alpha_{\sigma_N}}^{*}\sum_{\beta_{\sigma'_1}\ldots\beta_{\sigma'_N}}^{**}S(c,\sigma,\alpha_\sigma)S(c,\sigma',\beta_{\sigma'})\times$$
$$\times\prod_{i=1}^{N}P^{\alpha_{\sigma(i)}}_{\sigma(i)}R^{\sigma}_{1\ldots N}R^{\sigma'^{-1}}_{\sigma'(1\ldots N)}\prod_{i=1}^{N}P^{\beta_{\sigma'(i)}}_{\sigma'(i)}$$

$$= \sum_{\sigma \in \mathcal{S}_N} \sum_{\sigma' \in \mathcal{S}_N} \sum_{\alpha_{\sigma_1} \ldots \alpha_{\sigma_N}}^{*} \sum_{\beta_{\sigma'_1} \ldots \beta_{\sigma'_N}}^{**} S(c, \sigma, \alpha_\sigma) S(c, \sigma', \beta_{\sigma'}) \times$$

$$\times \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_{\sigma(i)}} R_{\sigma'(1 \ldots N)}^{\sigma'^{-1}\sigma} \prod_{i=1}^{N} P_{\sigma'(i)}^{\beta_{\sigma'(i)}} . \tag{3.19}$$

To evaluate the r.h.s., we examine the matrix element of the R-matrix

$$\left(R_{\sigma'(1 \ldots N)}^{\sigma'^{-1}\sigma} \right)_{\beta_{\sigma'(N)} \ldots \beta_{\sigma'(1)}}^{\alpha_{\sigma(N)} \ldots \alpha_{\sigma(1)}} . \tag{3.20}$$

Note that the sequence $\{\alpha_\sigma\}$ is non-decreasing and $\{\beta_{\sigma'}\}$ is non-increasing. Thus the non-vanishing condition of the matrix element (3.20) requires that α_σ and $\beta_{\sigma'}$ satisfy

$$\beta_{\sigma'(N)} = \alpha_{\sigma(1)}, \ldots, \beta_{\sigma'(1)} = \alpha_{\sigma(N)}. \tag{3.21}$$

One can verify [2] that (3.21) is fulfilled only if

$$\sigma'(N) = \sigma(1), \ldots, \sigma'(1) = \sigma(N) . \tag{3.22}$$

Let $\bar{\sigma}$ be the maximal element of the $\mathcal{S}_N$ which reverses the site labels

$$\bar{\sigma}(1, \ldots, N) = (N, \ldots, 1) . \tag{3.23}$$

Then from (3.22), we have

$$\sigma' = \sigma \bar{\sigma} . \tag{3.24}$$

Substituting (3.21) and (3.24) into (3.19), we have

$$F_{1 \ldots N} F_{1 \ldots N}^{*} = \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha_{\sigma_1} \ldots \alpha_{\sigma_N}}^{*} S(c, \sigma, \alpha_\sigma) S(c, \sigma, \alpha_\sigma) \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_{\sigma(i)}} R_{\sigma(N \ldots 1)}^{\bar{\sigma}} \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_{\sigma(i)}} . \tag{3.25}$$

The decomposition of $R^{\bar{\sigma}}$ in terms of elementary R-matrices is unique module GYBE. One reduces from (3.25) that FF^* is a diagonal matrix:

$$F_{1 \ldots N} F_{1 \ldots N}^{*} = \prod_{i<j} \Delta_{ij} , \tag{3.26}$$

where

$$[\Delta_{ij}]_{\alpha_i \alpha_j}^{\beta_i \beta_j} = \delta_{\alpha_i \beta_i} \delta_{\alpha_j \beta_j} \begin{cases} a_{ij} & \text{if } \alpha_i > \alpha_j \\ a_{ji} & \text{if } \alpha_i < \alpha_j \\ 1 & \text{if } \alpha_i = \alpha_j = 1, 2 \\ (1 + c_{ij})(1 + c_{ji}) & \text{if } \alpha_i = \alpha_j = 3 \end{cases} . \tag{3.27}$$

Therefore, the inverse of the F-matrix is given by

$$F_{1 \ldots N}^{-1} = F_{1 \ldots N}^{*} \prod_{i<j} \Delta_{ij}^{-1} . \tag{3.28}$$

4. The monodromy matrix in the F-basis

In the previous section, we see that the $gl(2|1)$ R-matrix factorizes in terms of the F-matrix and its inverse which we constructed explicitly. The column vectors of the inverse of the F-matrix form a set of basis on which $gl(2|1)$ acts. In this section, we study the generators of $gl(2|1)$ and the elements of the monodromy matrix in the F-basis.

4.1 $gl(2|1)$ generators in the F-basis

The N-site supersymmetric $gl(2|1)$ system has 4 simple generators: E^{12}, E^{21}, E^{23} and E^{32} with $E^{\gamma\,\gamma\pm1} = E^{\gamma\,\gamma\pm1}_{(1)} + \cdots + E^{\gamma\;\gamma\pm1}_{(N)}$, where $E^{\gamma\,\gamma\pm1}_{(k)}$ acts on the kth component of the tensor product space. Let $\tilde{E}^{\gamma\,\gamma\pm1}$ denote the corresponding simple generators in the F-basis: $\tilde{E}^{\gamma\,\gamma\pm1} = F_{1\ldots N}E^{\gamma\,\gamma\pm1}F^{-1}_{1\ldots N}$. We first derive the $\tilde{E}^{12}$. From the expressions of F and its inverse, we have

$$
\begin{aligned}
\tilde{E}^{12} &= F_{1\ldots N}E^{12}F^{-1}_{1\ldots N} \\
&= \sum_{\sigma,\sigma'\in\mathcal{S}_N}\ \sum^{*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\ \sum^{**}_{\beta_{\sigma'(1)}\ldots\beta_{\sigma'(N)}} S(c,\sigma,\alpha_\sigma)S(c,\sigma',\beta_{\sigma'})\,\times \\
&\quad\times\prod_{i=1}^{N}P^{\alpha_{\sigma(i)}}_{\sigma(i)}R^{\sigma}_{1\ldots N}E^{12}R^{\sigma'^{-1}}_{\sigma'(1\ldots N)}\prod_{i=1}^{N}P^{\beta_{\sigma'(i)}}_{\sigma'(i)}\prod_{i<j}\Delta^{-1}_{ij} \\
&= \sum_{\sigma,\sigma'\in\mathcal{S}_N}\ \sum^{*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\ \sum^{**}_{\beta_{\sigma'(1)}\ldots\beta_{\sigma'(N)}} S(c,\sigma,\alpha_\sigma)S(c,\sigma',\beta_{\sigma'})\,\times \\
&\quad\times\prod_{i=1}^{N}P^{\alpha_{\sigma(i)}}_{\sigma(i)}E^{12}R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)}\prod_{i=1}^{N}P^{\beta_{\sigma'(i)}}_{\sigma'(i)}\prod_{i<j}\Delta^{-1}_{ij} \qquad (4.1)\\
&= \sum_{\sigma,\sigma'\in\mathcal{S}_N}\ \sum_{k=1}^{N}E^{12}_{(\sigma(l))}\ \sum^{*}_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\ \sum^{**}_{\beta_{\sigma'(1)}\ldots\beta_{\sigma'(N)}} S(c,\sigma,\alpha_\sigma)S(c,\sigma',\beta_{\sigma'})\,\times \\
&\quad\times P^{\alpha_{\sigma(1)}=1}_{\sigma(1)}\ldots\left(P^{\alpha_{\sigma(l)}=1\to2}_{\sigma(l)=k}\right)\ldots P^{\alpha_{\sigma(N)}}_{\sigma(N)}R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)}\prod_{i=1}^{N}P^{\beta_{\sigma'(i)}}_{\sigma'(i)}\prod_{i<j}\Delta^{-1}_{ij}, \qquad (4.2)
\end{aligned}
$$

where in (4.1), we have used $[E^{\gamma\,\gamma\pm1}, R^{\sigma}_{1\ldots N}] = 0$. The element of $R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)}$ between

$$P^{\alpha_{\sigma(1)}=1}_{\sigma(1)}\cdots\left(P^{\alpha_{\sigma(l)}=1\to2}_{\sigma(l)=k}\right)\cdots P^{\alpha_{\sigma(N)}}_{\sigma(N)}$$

and

$$P^{\beta_{\sigma'(N)}}_{\sigma'(N)}\cdots P^{\beta_{\sigma'(1)}}_{\sigma'(1)}$$

is denoted as

$$\left(R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)}\right)^{\overset{\sigma(N)}{\alpha_{\sigma(N)}}\ldots\overset{\sigma(l)=k}{1\to2}\ldots\overset{\sigma(1)}{1}}_{\beta_{\sigma'(N)}\ldots\beta_{\sigma'(1)}}. \qquad (4.3)$$

We call the sequence $\{\alpha_{\sigma(l)}\}$ *normal* if it is arranged according to the rules in (3.7), otherwise, we call it *abnormal*.

It is now convenient for us to discuss the non-vanishing condition of the R-matrix element (4.3). Comparing (4.3) with (3.20), we find that the difference between them lies in the kth site. Because the group label in the kth space has been changed, the sequence $\{\alpha_\sigma\}$ is now a abnormal sequence. However, it can be permuted to the normal sequence by some permutation $\hat{\sigma}_k$. Namely, $\alpha_{1\to 2}$ in the abnormal sequence can be moved to a suitable position by using the permutation $\hat{\sigma}_k$ according to rules in (3.7). (It is easy to verify that $\hat{\sigma}_k$ is unique by using (3.7).) Thus, by procedure similar to that in the previous section, we find that when

$$\sigma' = \hat{\sigma}_k \sigma \bar{\sigma} \qquad \text{and} \qquad \beta_{\sigma'(N)} = \alpha_{\sigma(1)}, \ldots, \beta_{\sigma'(1)} = \alpha_{\sigma(N)}\,, \tag{4.4}$$

the R-matrix element (4.3) is non-vanishing.

Because the non-zero condition of the elementary R-matrix element $R_{ij}^{i'j'}$ is $i+j = i'+j'$, the following R-matrix elements

$$\left(R^{\sigma'^{-1}\sigma}_{\sigma'(1\ldots N)}\right)^{\overset{\sigma(N)}{\alpha_{\sigma(N)}}\ldots\overset{\sigma(l)=k}{1}\ldots\overset{\sigma(n)}{1\to 2}\ldots\overset{\sigma(1)}{1}}_{\beta_{\sigma'(N)}\ldots\beta_{\sigma'(1)}} \tag{4.5}$$

with $1 \le n < l$ are also non-vanishing.

Therefore, (4.2) becomes

$$\begin{aligned}
\tilde{E}^{12} = & \sum_{\sigma\in\mathcal{S}_N}\sum_{k=1}^{N}\sum_{\alpha_{\sigma_1}\ldots\alpha_{\sigma_N}}^{*} S(c,\sigma,\alpha_\sigma)S(c,\hat{\sigma}_k\sigma,\alpha_{\hat{\sigma}_k\sigma}) \times \\
& \times \Big[E^{12}_{(\sigma(l))}P^{\alpha_{\sigma(1)}=1}_{\sigma(1)}\cdots P^{\alpha_{\sigma(l)}=1\to 2}_{\sigma(l)=k}\cdots P^{\alpha_{\sigma(N)}}_{\sigma(N)} + \cdots + \\
& \quad + E^{12}_{(\sigma(n))}P^{\alpha_{\sigma(1)}=1}_{\sigma(1)}\cdots P^{\alpha_{\sigma(n)}=1\to 2}_{\sigma(n)}\cdots P^{\alpha_{\sigma(l)}=1}_{\sigma(l)=k}\cdots P^{\alpha_{\sigma(N)}}_{\sigma(N)} + \cdots + \\
& \quad + E^{12}_{(\sigma(1))}P^{\alpha_{\sigma(1)}=1\to 2}_{\sigma(1)}\ldots P^{\alpha_{\sigma(l)}=1}_{\sigma(l)=k}\ldots P^{\alpha_{\sigma(N)}}_{\sigma(N)}\Big] \times \\
& \times R^{\bar{\sigma}\sigma^{-1}\hat{\sigma}_k^{-1}\sigma}_{\hat{\sigma}_k\sigma(N\ldots 1)}\prod_{i=1}^{N}P^{\alpha_{\hat{\sigma}_k\sigma(i)}}_{\hat{\sigma}_k\sigma(i)}\prod_{i<j}\Delta_{ij}^{-1}\,, \qquad (4.6)\\
= & \sum_{k=1}^{N} E^{12}_{(k)} \otimes_{j\neq k} G^{12}_{(j)}(k,j)\,, \qquad (4.7)
\end{aligned}$$

where $\hat{\sigma}_k$ is the element of $\mathcal{S}_N$ which permutes the first abnormal sequence in the square bracket of (4.6) to normal sequence, and

$$G^{12}(k,j) = \mathrm{diag}(a_{kj}^{-1}, 1, 1)\,. \tag{4.8}$$

Similarly, we obtain the expressions of other three simple generators in the F-basis:

$$\tilde{E}^{\gamma\,\gamma\pm 1} = \sum_{i=1}^{N} E^{\gamma\,\gamma\pm 1}_{(i)} \otimes_{j\neq i} G^{\gamma,\gamma\pm 1}_{(j)}(i,j) \tag{4.9}$$

with

$$\begin{aligned}
G^{21}(i,j) &= \mathrm{diag}(1, a_{ji}^{-1}, 1)\,, \\
G^{23}(i,j) &= \mathrm{diag}\left(1, a_{ij}^{-1}, (2a_{ij})^{-1}\right), \\
G^{32}(i,j) &= \mathrm{diag}(1,1,2)\,.
\end{aligned} \tag{4.10}$$

With the help of the simple generators, the non-simple generators $\tilde{E}^{13}$ and $\tilde{E}^{31}$ of $gl(2|1)$ can be obtained by the commutation relations,

$$\tilde{E}^{13} = [\tilde{E}^{12}, \tilde{E}^{23}]\,, \qquad \tilde{E}^{31} = [\tilde{E}^{32}, \tilde{E}^{21}]\,. \tag{4.11}$$

Substituting (4.7) and (4.9) into (4.11), we obtain

$$\begin{aligned}
\tilde{E}^{13} &= \sum_{i=1}^{N} E_{(i)}^{13} \otimes_{j\neq i} \mathrm{diag}\left(a_{ij}^{-1}, a_{ij}^{-1}, (2a_{ij})^{-1}\right)_{(j)} + \\
&\quad + \sum_{i\neq j=1}^{N} \frac{\eta}{z_j - z_i} E_{(i)}^{12} \otimes E_{(j)}^{23} \otimes_{k\neq i,j} \mathrm{diag}\left(a_{ik}^{-1}, a_{jk}^{-1}, (2a_{jk})^{-1}\right)_{(k)}, \\
\tilde{E}^{31} &= \sum_{i=1}^{N} E_{(i)}^{31} \otimes_{j\neq i} \mathrm{diag}\left(1, a_{ji}^{-1}, 2\right)_{(j)} + \\
&\quad + \sum_{i\neq j=1}^{N} \frac{\eta}{z_i - z_j} E_{(i)}^{32} \otimes E_{(j)}^{21} \otimes_{k\neq i,j} \mathrm{diag}\left(1, a_{kj}^{-1}, 2\right)_{(k)}\,.
\end{aligned} \tag{4.12}$$

4.2 Elements of the monodromy matrix in the F-basis

In the F-basis, the monodromy matrix $T(u)$, (2.8), becomes

$$\tilde{T}(u) \equiv F_{1\ldots N} T(u) F_{1\ldots N}^{-1} = \begin{pmatrix} \tilde{A}_{11}(u) & \tilde{A}_{12}(u) & \tilde{B}_1(u) \\ \tilde{A}_{21}(u) & \tilde{A}_{22}(u) & \tilde{B}_2(u) \\ \tilde{C}_1(u) & \tilde{C}_2(u) & \tilde{D}(u) \end{pmatrix}. \tag{4.13}$$

We first study the diagonal element $\tilde{D}(u)$. Acting the F-matrix on $D(u)$, we have

$$\begin{aligned}
F_{1\ldots N} T^{33} &= \sum_{\sigma\in\mathcal{S}_N} \sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}^{*} S(c,\sigma,\alpha_\sigma) \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_\sigma} R_{1\ldots N}^{\sigma} P_0^3 T_{0,1\ldots N} P_0^3 \\
&= \sum_{\sigma\in\mathcal{S}_N} \sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}^{*} S(c,\sigma,\alpha_\sigma) \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_\sigma} P_0^3 T_{0,\sigma(1\ldots N)} P_0^3 R_{1\ldots N}^{\sigma}.
\end{aligned} \tag{4.14}$$

Following [2], we can split the sum $\sum^*$ according to the number of the occurrences of the index 3.

$$\begin{aligned}
F_{1\ldots N} T^{33} &= \sum_{\sigma\in\mathcal{S}_N} \sum_{k=0}^{N} \sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}^{*} S(c,\sigma,\alpha_\sigma) \prod_{j=N-k+1}^{N} \delta_{\alpha_{\sigma(j)},3} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \times \\
&\quad \times \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} P_0^3 T_{0,\sigma(1\ldots N)} P_0^3 R_{1\ldots N}^{\sigma}\,.
\end{aligned} \tag{4.15}$$

Consider the prefactor of $R^{\sigma}_{1\ldots N}$. We have

$$
\begin{aligned}
&\prod_{j=1}^{N-k} P^{\alpha_{\sigma(j)}}_{\sigma(j)} \prod_{j=N-k+1}^{N} P^3_{\sigma(j)} P^3_0 T_{0,\sigma(1\ldots N)} P^3_0 = \\
&\quad = \prod_{j=1}^{N-k} P^{\alpha_{\sigma(j)}}_{\sigma(j)} \prod_{j=N-k+1}^{N} \left(R_{0\,\sigma(j)}\right)^{33}_{33} P^3_0 T_{0,\sigma(1\ldots N-k)} P^3_0 \prod_{j=N-k+1}^{N} P^3_{\sigma(j)} \\
&\quad = \prod_{i=N-k+1}^{N} c_{0\,\sigma(i)} \prod_{j=1}^{N-k} P^{\alpha_{\sigma(j)}}_{\sigma(j)} P^3_0 T_{0,\sigma(1\ldots N-k)} P^3_0 \prod_{j=N-k+1}^{N} P^3_{\sigma(j)} \\
&\quad = \prod_{i=N-k+1}^{N} c_{0\,\sigma(i)} \prod_{i=1}^{N-k} \left(R_{0\,\sigma(i)}\right)^{3\alpha_{\sigma(i)}}_{3\alpha_{\sigma(i)}} \prod_{j=1}^{N-k} P^{\alpha_{\sigma(j)}}_{\sigma(j)} \prod_{j=N-k+1}^{N} P^3_{\sigma(j)}\,, \\
&\quad = \prod_{i=N-k+1}^{N} c_{0\,\sigma(i)} \prod_{i=1}^{N-k} a_{0\,\sigma(i)} \prod_{j=1}^{N-k} P^{\alpha_{\sigma(j)}}_{\sigma(j)} \prod_{j=N-k+1}^{N} P^3_{\sigma(j)}\,,
\end{aligned} \tag{4.16}
$$

where $c_{0i} = c(u, z_i)$, $a_{0i} = a(u, z_i)$. Substituting (4.16) into (4.15), we have

$$F_{1\ldots N} T^{33} = \otimes_{i=1}^{N} \operatorname{diag}\left(a_{0i}, a_{0i}, c_{0i}\right)_{(i)} F_{1\ldots N}\,. \tag{4.17}$$

Therefore,

$$\tilde{D}(u) = \tilde{T}^{33}(u) = \otimes_{i=1}^{N} \operatorname{diag}\left(a_{0i}, a_{0i}, c_{0i}\right)_{(i)}\,. \tag{4.18}$$

The other elements of the monodromy matrix can then be obtained as follows:

$$\tilde{T}^{3\alpha} = [\tilde{E}^{\alpha\,3}, \tilde{T}^{33}]\,, \qquad \tilde{T}^{\alpha 3} = [\tilde{E}^{3\,\alpha}, \tilde{T}^{33}]\,, \qquad (\alpha = 1, 2)\,, \tag{4.19}$$

which follows from the $gl(2|1)$ invariance of the R-matrix, i.e. in terms of the monodromy matrix,

$$[\tilde{T}(u), \tilde{E}^{\alpha\beta}_{(0)} + \tilde{E}^{\alpha\beta}] = 0\,. \tag{4.20}$$

Substituting $\tilde{E}^{\alpha\,3}$, $\tilde{E}^{3\,\alpha}$ and $\tilde{T}^{33}$ into the above relations yields

$$\tilde{T}^{32} = -\sum_{i=1}^{N} b_{0i} E^{23}_{(i)} \otimes_{j\neq i} \operatorname{diag}\left(a_{0j}, a_{0j} a^{-1}_{ij}, c_{0j} (2a_{ij})^{-1}\right)_{(j)}\,, \tag{4.21}$$

$$\tilde{T}^{23} = \sum_{i=1}^{N} b_{0i} E^{32}_{(i)} \otimes_{j\neq i} \operatorname{diag}\left(a_{0j}, a_{0j}, 2c_{0j}\right)_{(j)}\,, \tag{4.22}$$

$$
\begin{aligned}
\tilde{T}^{31} &= -\sum_{i=1}^{N} b_{0i} E^{13}_{(i)} \otimes_{j\neq i} \operatorname{diag}\left(a_{0j} a^{-1}_{ij}, a_{0j} a^{-1}_{ij}, c_{0j} (2a_{ij})^{-1}\right)_{(j)} - \\
&\quad - \sum_{i\neq j=1}^{N} \frac{a_{0i} b_{0j} \eta}{z_j - z_i} E^{12}_{(i)} \otimes E^{23}_{(j)} \otimes_{k\neq i,j} \operatorname{diag}\left(a_{0k} a^{-1}_{ik}, a_{0k} a^{-1}_{jk}, c_{0k} (2a_{jk})^{-1}\right)_{(k)}\,, \\
\tilde{T}^{13} &= \sum_{i=1}^{N} b_{0i} E^{31}_{(i)} \otimes_{j\neq i} \operatorname{diag}\left(a_{0j}, a_{0j} a^{-1}_{ji}, 2c_{0j}\right)_{(j)} + \\
&\quad + \sum_{i\neq j=1}^{N} \frac{a_{0i} b_{0j} \eta}{z_i - z_j} E^{32}_{(i)} \otimes E^{21}_{(j)} \otimes_{k\neq i,j} \operatorname{diag}\left(a_{0k}, a_{0k} a^{-1}_{kj}, 2c_{0k}\right)_{(k)}\,.
\end{aligned} \tag{4.23}
$$

Here, b_{0j} stands for $b(u, z_j)$.

5. Bethe vectors in the F-basis

Having obtained the creation operators of the $gl(2|1)$ monodromy matrix in the F-basis, we are now in the position to study the Bethe vectors in this basis. Acting the F-matrix on the pseudo-vacuum state (A.1), we obtain

$$F_{1\ldots N}|0\rangle = \prod_{i<j}^{N}(1+c_{ij})|0\rangle \equiv s(c)|0\rangle\,. \tag{5.1}$$

Therefore, the $gl(2|1)$ Bethe vector (A.2) in the F-basis can be written as

$$\begin{aligned}\tilde{\Phi}_N(v_1,\ldots,v_n) &\equiv F_{1\ldots N}\Phi_N(v_1,\ldots,v_n)\\ &= s(c)\sum_{d_1\ldots d_n}(\phi_n^{(1)})^{d_1\ldots d_n}\tilde{C}_{d_1}(v_1)\ldots\tilde{C}_{d_n}(v_n)|0\rangle\,,\end{aligned} \tag{5.2}$$

where $d_i=1,2$ and $(\phi_n^{(1)})^{d_1\ldots d_n}$ is the coefficients of the nested Bethe vector $\phi_n^{(1)}(v_1^{(1)},\ldots,v_m^{(1)})$, (A.10), associated with the $gl(2)$ transfer matrix $t^{(1)}(u)$ with inhomogeneous parameters $v_1,\ldots,v_n$. The c-number coefficient $(\phi_n^{(1)})^{d_1\ldots d_n}$ has to be evaluated in the original basis, not in the F-basis.

Let us first compute the F-transformed nested Bethe vector. Denote by $F^{(1)}$ the $gl(2)$ F-matrix. Applying $F^{(1)}$ to the nested Bethe vector $\phi_n^{(1)}$, we obtain

$$\begin{aligned}\tilde{\phi}_n^{(1)}(v_1^{(1)},\ldots,v_m^{(1)}) &\equiv F_{1\ldots n}^{(1)}\phi^{(1)}(v_1^{(1)},\ldots,v_m^{(1)})\\ &= \tilde{C}^{(1)}(v_1^{(1)})\tilde{C}^{(1)}(v_2^{(1)})\ldots\tilde{C}^{(1)}(v_m^{(1)})|0\rangle^{(1)}\,,\end{aligned} \tag{5.3}$$

where the nested pseudo-vacuum state $|0\rangle^{(1)}$ is invariant under the action of the $gl(2)$ F-matrix. From [1], the F-transformed $gl(2)$ creation operator $\tilde{C}^{(1)}$ is given by

$$\tilde{C}^{(1)}(v^{(1)}) = \sum_{i=1}^{n} b(v^{(1)},v_i)\sigma_{(i)}^{+}\otimes_{j\neq i}\begin{pmatrix} a(v^{(1)},v_j)a_{ij}^{-1} & 0\\ 0 & 1\end{pmatrix}. \tag{5.4}$$

Substituting $\tilde{C}^{(1)}(v)$ into (5.3), we obtain

$$\begin{aligned}\tilde{\phi}_n^{(1)}(v_1^{(1)},\ldots,v_m^{(1)}) &= \tilde{C}^{(1)}(v_1^{(1)})\ldots\tilde{C}^{(1)}(v_m^{(1)})\,|0\rangle^{(1)}\\ &= \sum_{i_1<\ldots<i_m} B_m^{(1)}(v_1^{(1)},\ldots,v_m^{(1)}|v_{i_1},\ldots,v_{i_m})\sigma_{(i_1)}^{+}\cdots\sigma_{(i_m)}^{+}\,|0\rangle^{(1)}\,,\end{aligned} \tag{5.5}$$

where

$$B_m^{(1)}(v_1^{(1)},\ldots,v_m^{(1)}|v_1,\ldots,v_m) = \sum_{\sigma\in S_m}\prod_{k=1}^{m} b(v_k^{(1)},v_{\sigma(k)})\prod_{l=k+1}^{m}\frac{a(v_k^{(1)},v_{\sigma(l)})}{a(v_{\sigma(k)},v_{\sigma(l)})}\,. \tag{5.6}$$

Now back to the $gl(2|1)$ Bethe vector (5.2). As is shown in Appendix B, the Bethe vector is invariant (module overall factor) under the exchange of arbitrary spectral parameters:

$$\tilde{\Phi}_N(v_{\sigma(1)},\ldots,v_{\sigma(n)}) = \mathrm{sign}(\sigma)c_{1\ldots n}^{\sigma}\tilde{\Phi}_N(v_1,\ldots,v_n)\,. \tag{5.7}$$

This enable one to concentrate on a particularly simple term in the sum (5.2) of the following form with p_1 number of $d_i = 1$ and $n - p_1$ number of $d_j = 2$

$$\tilde{C}_1(v_1)\cdots\tilde{C}_1(v_{p_1})\tilde{C}_2(v_{p_1+1})\cdots\tilde{C}_2(v_n). \tag{5.8}$$

In the F-basis, the commutation relation between $C_i(v)$ and $C_j(u)$, in (A.4), becomes

$$\tilde{C}_i(v)\tilde{C}_j(u) = -\frac{c(u,v)}{a(u,v)}\tilde{C}_j(u)\tilde{C}_i(v) - \frac{b(u,v)}{a(u,v)}\tilde{C}_j(v)\tilde{C}_i(u)\,. \tag{5.9}$$

Then using (5.9), all $\tilde{C}_1$'s in (5.8) can be moved to the right of all $\tilde{C}_2$'s, yielding

$$\begin{aligned}&\tilde{C}_1(v_1)\ldots\tilde{C}_1(v_{p_1})\tilde{C}_2(v_{p_1+1})\ldots\tilde{C}_2(v_n) = \\ &\quad = g(v_1,\ldots,v_n)\tilde{C}_2(v_{p_1+1})\ldots\tilde{C}_2(v_n)\tilde{C}_1(v_1)\ldots\tilde{C}_1(v_{p_1}) + \cdots\,,\end{aligned} \tag{5.10}$$

where $g(v_1,\ldots,v_n) = \prod_{k=1}^{p_1}\prod_{l=p_1+1}^{n}(-c(v_l,v_k)/a(v_l,v_k))$ is the contribution from the first term of (5.9) and "..." stands for the other terms contributed by the second term of (5.9). It is easy to see that the other terms have the form

$$\tilde{C}_2(v_{\sigma(p_1+1)})\ldots\tilde{C}_2(v_{\sigma(n)})\tilde{C}_1(v_{\sigma(1)})\ldots\tilde{C}_1(v_{\sigma(p_1)}) \tag{5.11}$$

with $\sigma \in \mathcal{S}_n$. Substituting (5.10) into the Bethe vector (5.2), we obtain

$$\begin{aligned}\tilde{\Phi}_N^{p_1}(v_1,\ldots,v_n) \;=\; & s(c)(\phi_n^{(1)})^{11\ldots12\ldots2}\prod_{k=1}^{p_1}\prod_{l=p_1+1}^{n}\left(-\frac{c(v_l,v_k)}{a(v_l,v_k)}\right)\times \\ & \times\tilde{C}_2(v_{p_1+1})\ldots\tilde{C}_2(v_n)\tilde{C}_1(v_1)\ldots\tilde{C}_1(v_{p_1})|0\rangle + \cdots\,,\end{aligned} \tag{5.12}$$

where and below, we have used the up-index p_1 to denote the Bethe vector corresponding to the quantum number p_1. All other terms in (5.12) (denoted as "...") are to be obtained from the first term by the permutation (exchange) symmetry. Thus (see appendix C for the $n = 2$ case),

$$\begin{aligned}&\tilde{\Phi}_N^{p_1}(v_1,\ldots,v_n) = \\ &\quad = \frac{s(c)}{p_1!(n-p_1)!}\sum_{\sigma\in\mathcal{S}_n}\mathrm{sign}(\sigma)(c^\sigma_{1\ldots n})^{-1}(\phi_n^{(1),\sigma})^{11\ldots12\ldots2}\prod_{k=1}^{p_1}\prod_{l=p_1+1}^{n}\left(-\frac{c(v_{\sigma(l)},v_{\sigma(k)})}{a(v_{\sigma(l)},v_{\sigma(k)})}\right)\times \\ &\quad\times\,\tilde{C}_2(v_{\sigma(p_1+1)})\ldots\tilde{C}_2(v_{\sigma(n)})\tilde{C}_1(v_{\sigma(1)})\ldots\tilde{C}_1(v_{\sigma(p_1)})\,|0\rangle\,,\end{aligned} \tag{5.13}$$

where $(\phi_n^{(1),\sigma})^{11\ldots12\ldots2} \equiv (\hat{f}_\sigma\phi_n^{(1)})^{11\ldots12\ldots2}$ with $\hat{f}_\sigma$ defined by (B.1) in the appendix B.

We now show that $(\phi_n^{(1)})^{1\ldots12\ldots2}$ in (5.13), which has to be evaluated in the original basis, is invariant under the action of the $gl(2)$ F-matrix, i.e.

$$(\phi_n^{(1)})^{11\ldots12\ldots2} = (\tilde{\phi}_n^{(1)})^{11\ldots12\ldots2}\,, \tag{5.14}$$

so that it can be expressed in the form of (5.6).

Write the nested pseudo-vacuum vector in (A.10) as

$$|0\rangle^{(1)} \equiv |2\cdots2\rangle^{(1)}\,, \tag{5.15}$$

where the number of 2 is n. Then the nested Bethe vector (A.10) can be rewritten as

$$\phi_n^{(1)}\left(v_1^{(1)}\dots v_{p_1}^{(1)}\right) \equiv |\phi_n^{(1)}\rangle = \sum_{d_1\dots d_n} (\phi_n^{(1)})^{d_1\dots d_n} |d_1\dots d_n\rangle^{(1)} . \tag{5.16}$$

Acting the $gl(2)$ F-matrix $F^{(1)}$ from left on the above equation, we have

$$\tilde{\phi}_n^{(1)}\left(v_1^{(1)}\dots v_{p_1}^{(1)}\right) \equiv |\tilde{\phi}_n^{(1)}\rangle = F^{(1)}|\phi_n^{(1)}\rangle = \sum_{d_1\dots d_n} (\tilde{\phi}_n^{(1)})^{d_1\dots d_n} |d_1\dots d_n\rangle^{(1)} . \tag{5.17}$$

It follows that

$$\begin{aligned}
(\tilde{\phi}_n^{(1)})^{1\dots 12\dots 2} &= \langle 1\dots 12\dots 2|\tilde{\phi}_n^{(1)}\rangle = \rangle 1\dots 12\dots 2|F^{(1)}|\phi_n^{(1)}\rangle \\
&= \langle 1\dots 12\dots 2| \sum_{\sigma\in\mathcal{S}_n} \sum_{\alpha_{\sigma(1)}\dots\alpha_{\sigma(n)}}^{*} \prod_{j=1}^{n} P_{\sigma(j)}^{\alpha_{\sigma(j)}} R_{1\dots n}^{\sigma} |\phi_n^{(1)}\rangle && (5.18)\\
&= \langle 1\dots 12\dots 2| \left. \left\{ \sum_{\alpha_{\sigma(1)}\dots\alpha_{\sigma(n)}}^{*} \prod_{j=1}^{n} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \right\}\right|_{\sigma=id} R_{1\dots n}^{\sigma=id} |\phi_n^{(1)}\rangle && (5.19)\\
&= \langle 1\dots 12\dots 2|\phi_n^{(1)}\rangle = (\phi_n^{(1)})^{1\dots 12\dots 2} , && (5.20)
\end{aligned}$$

Summarizing, we propose the following form of the $gl(2|1)$ Bethe vector

$$\begin{aligned}
&\tilde{\Phi}_N^{p_1}(v_1,\dots,v_n) = \\
&= \frac{s(c)}{p_1!(n-p_1)!} \sum_{\sigma\in\mathcal{S}_n} \mathrm{sign}(\sigma) \left((c_{1\dots n}^{\sigma})^{-1} B_{p_1}^{(1)}(v_1^{(1)},\dots,v_{p_1}^{(1)}|v_{\sigma(1)},\dots,v_{\sigma(p_1)} \right) \times \\
&\quad \times \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{n} \left(-\frac{c(v_{\sigma(l)},v_{\sigma(k)})}{a(v_{\sigma(l)},v_{\sigma(k)})} \right) \tilde{C}_2(v_{\sigma(p_1+1)})\dots\tilde{C}_2(v_{\sigma(n)}) \times \\
&\quad \times \tilde{C}_1(v_{\sigma(1)})\dots\tilde{C}_1(v_{\sigma(p_1)})\,|0\rangle . && (5.21)
\end{aligned}$$

Substituting (4.21) and (4.23) into the above relation, we finally obtain

$$\begin{aligned}
\tilde{\Phi}_N^{p_1}(v_1,\dots,v_n) &= \frac{s(c)}{p_1!(n-p_1)!} \sum_{i_1<\dots<i_n} B_{n,p_1}\left(v_1,\dots,v_n;v_1^{(1)},\dots,v_{p_1}^{(1)}|z_{i_1},\dots,z_{i_n}\right) \times \\
&\quad \times \prod_{j=i_{p_1+1}}^{i_n} E_{(j)}^{23} \prod_{j=i_1}^{i_{p_1}} E_{(j)}^{13}\,|0\rangle , && (5.22)
\end{aligned}$$

where

$$\begin{aligned}
&B_{n,p_1}(v_1,\dots,v_n;v_1^{(1)},\dots,v_{p_1}^{(1)}|z_{i_1},\dots,z_{i_n}) = \\
&= \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)(c_{1\dots n}^{\sigma})^{-1} \prod_{k=1}^{p_1} \prod_{l=p_1+1}^{n} \left(-\frac{c(v_{\sigma(l)},v_{\sigma(k)})a(v_{\sigma(l)},z_{i_k})}{a(v_{\sigma(l)},v_{\sigma(k)})} \right) \times \\
&\quad \times B_{n-p_1}^{*}(v_{\sigma(p_1+1)},\dots,v_{\sigma(n)}|z_{i_{p_1+1}},\dots,z_{i_n}) \times \\
&\quad \times B_{p_1}^{(1)}(v_1^{(1)},\dots,v_{p_1}^{(1)}|v_{\sigma(1)},\dots,v_{\sigma(p_1)}) B_{p_1}^{*}(v_{\sigma(1)},\dots,v_{\sigma(p_1)}|z_{i_1},\dots,z_{i_{p_1}}) && (5.23)
\end{aligned}$$

with

$$B_p^*(v_1,\ldots,v_p|z_1,\ldots,z_p) = \tag{5.24}$$
$$= \sum_{\sigma\in\mathcal{S}_p} \mathrm{sign}(\sigma) \prod_{m=1}^{p} (-b(v_m, z_{\sigma(m)})) \prod_{j\neq\sigma(1),\ldots,\sigma(m)}^{N} \frac{c(v_m,z_j)}{2a(z_{\sigma(m)},z_j)} \prod_{l=m+1}^{p} \frac{a(v_m,z_{\sigma(l)})}{a(z_{\sigma(m)},z_{\sigma(l)})}\,.$$

6. Discussions

In this paper, we have constructed the factorizing F-matrices for the supersymmetric t-J model. In the basis provided by the F-matrix (the F-basis), the monodromy matrix and the creation operators take completely symmetric forms. We moreover have obtained a simple representation of the Bethe vector of the system.

Authors in [16] solved the quantum inverse problem of the supersymmetric t-J model in the original basis. Namely they reconstructed the local operators (E^{ij}) in terms of operators figuring in the $gl(2|1)$ monodromy matrix. This together with the results of the present paper in the F-basis should enable one to get the exact representations of form factors and correlation functions of the supersymmetric t-J model. These are under investigation and results will be reported elsewhere.

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A. The nested Bethe ansatz for the $gl(2|1)$ model

In this Appendix, we recall the nested Bethe ansatz method [11, 12]. The hamiltonian (2.11) can be exactly diagonalized by using the nested Bethe ansatz method. Define the pseudo-vacuum state

$$|0\rangle_k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |0\rangle = \otimes_{k=1}^{N} |0\rangle_k \tag{A.1}$$

and the Bethe vector

$$\Phi_N(v_1,\ldots,v_n) = \sum_{d_1,\ldots,d_n} (\phi_n^{(1)})^{d_1\ldots d_n} C_{d_1}(v_1) C_{d_2}(v_2)\ldots C_{d_n}(v_n)|0\rangle, \tag{A.2}$$

where $(\phi_n^{(1)})^{d_1\ldots d_n}$ is a function of the spectral parameters v_j.

Applying the quantum operators A_{ij}, B_i, C_i and D to the pseudo-vacuum state, we obtain

$$D(u)|0\rangle = \prod_{i=1}^{N} c(u,z_i)|0\rangle\,, \qquad A_{ij}(u)|0\rangle = \delta_{ij} \prod_{k=1}^{N} a(u,z_k)|0\rangle,$$
$$B_i|0\rangle = 0\,. \tag{A.3}$$

From the GYBE (2.9), one obtains the following commutation relations

$$\begin{aligned} C_i(u)C_j(v) &= -\sum_{k,l} \frac{r(u,v)_{ji}^{kl}}{c(u,v)} C_k(v)C_l(u)\,, \\ D(u)C_j(v) &= \frac{c(v,u)}{a(v,u)} C_j(v)D(u) + \frac{b(v,u)}{a(v,u)} C_j(u)D(v)\,, \\ A_{ij}(u)C_k(v) &= \sum_{m,l} \frac{r(u,v)_{jk}^{ml}}{a(u,v)} C_l(v)A_{im}(u) - \frac{b(u,v)}{a(u,v)} C_j(u)A_{ik}(v)\,, \end{aligned} \tag{A.4}$$

where

$$r(u,v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(u,v) & b(u,v) & 0 \\ 0 & b(u,v) & a(u,v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.5}$$

is the $gl(2)$ R-matrix acting on the tensor product of the 2-dimensional representation of $gl(2)$. With the help of the commutation relations (A.4), we have the action of $D(u)$ on the Bethe vector

$$\begin{aligned} &D(u) \sum_{d_1,\ldots,d_n} (\phi_n^{(1)})^{d_1\ldots d_n} C_{d_1}(v_1)C_{d_2}(v_2)\ldots C_{d_n}(v_n)|0\rangle \\ &= \prod_{i=1}^{N} c(u,z_i) \prod_{j=1}^{n} \frac{c(v_j,u)}{a(v_j,u)} \sum_{d_1,\ldots,d_n} (\phi_n^{(1)})^{d_1\ldots d_n} C_{d_1}(v_1)\ldots C_{d_n}(v_n)|0\rangle + u.t.\,, \end{aligned} \tag{A.6}$$

Similarly, the action of A_{aa} $(a = 1,2)$ on the Bethe vector gives rise to

$$\begin{aligned} &A_{aa}(u) \sum_{d_1,\ldots,d_n} (\phi_n^{(1)})^{d_1\ldots d_n} C_{d_1}(v_1)\ldots C_{d_n}(v_n)|0\rangle \\ &\quad = \prod_{i=1}^{N} a(u,z_i) \prod_{j=1}^{n} \frac{1}{a(u,v_j)} \sum_{d_1,\ldots,d_n} (\phi_n^{(1)})^{d_1\ldots d_n} C_{q_1}(v_1)\ldots C_{q_n}(v_n)|0\rangle \times \\ &\qquad \times\, r(u,v_1)_{ad_1}^{c_1q_1}\, r(u,v_1)_{c_1d_2}^{c_2q_2} \ldots r(u,v_n)_{c_{n-1}d_n}^{a\ \ q_n} + u.t. \\ &\equiv \prod_{i=1}^{N} a(u,z_i) \prod_{j=1}^{n} \frac{1}{a(u,v_j)} C_{q_1}(v_1)\ldots C_{q_n}(v_n)|0\rangle \times \\ &\qquad \times \sum_{d_1,\ldots,d_n} t^{(1)}(u)_{d_1\ldots d_n}^{q_1\ldots q_n} (\phi_n^{(1)})^{d_1\ldots d_n} + u.t.\,, \end{aligned} \tag{A.7}$$

where

$$t^{(1)}(u) = tr_0 T^{(1)}(u) \tag{A.8}$$

is the nested transfer matrix with

$$\begin{aligned} T^{(1)}(u) &\equiv r_n(u,v_n)\ldots r_1(u,v_1) \\ &= L_n^{(1)}(u,v_n)\ldots L_i^{(1)}(u,v_i)\ldots L_1^{(1)}(u,v_1) \\ &= \begin{pmatrix} A^{(1)}(u) & B^{(1)}(u) \\ C^{(1)}(u) & D^{(1)}(u) \end{pmatrix} \end{aligned} \tag{A.9}$$

being the nested monodromy matrix. (A.7) results in an eigenvector of $A_{aa}(u)$ if

$$\sum_{d_1,\ldots,d_n} t^{(1)}(u)^{q_1\ldots q_n}_{d_1\ldots d_n}(\phi_n^{(1)})^{d_1\ldots d_n} = \varepsilon^{(1)}(u)(\phi_n^{(1)})^{q_1\ldots q_n}\,.$$

This is nothing but a Bethe ansatz problem for $gl(2)$ chain of length n with the inhomogeneities now given by the parameters $v_1,\ldots,v_n$ of the $gl(2|1)$ problem. This inspires one to define $\phi_n^{(1)}$ as

$$\phi_n^{(1)}\left(v_1^{(1)},\ldots,v_m^{(1)}\right) = C^{(1)}(v_1^{(1)})C^{(1)}(v_2^{(1)})\ldots C^{(1)}(v_m^{(1)})|0\rangle^{(1)}\,, \tag{A.10}$$

where $|0\rangle^{(1)}$ is the 2-dimensional pseudo-vacuum

$$|0\rangle_k^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |0\rangle^{(1)} = \otimes_{k=1}^n |0\rangle_k^{(1)}\,. \tag{A.11}$$

Then $\phi_n^{(1)}$ spans a subspace of the space spanned by Φ.

The action of the nested monodromy matrix elements on the nested pseudo-vacuum (A.11) read

$$\begin{aligned} A^{(1)}(u)|0\rangle^{(1)} &= \prod_{i=1}^n a(u,v_i)|0\rangle^{(1)}\,, \qquad D^{(1)}(u)|0\rangle^{(1)} = |0\rangle^{(1)}\,, \\ B^{(1)}(u)|0\rangle^{(1)} &= 0\,. \end{aligned} \tag{A.12}$$

The commutation relations between the elements of the nested monodromy matrix are given

$$\begin{aligned} A^{(1)}(u)C^{(1)}(v) &= \frac{1}{a(u,v)}C^{(1)}(v)A^{(1)}(u) - \frac{b(u,v)}{a(u,v)}C^{(1)}(u)(u)A^{(1)}(v)\,, \\ D^{(1)}(u)C^{(1)}(v) &= \frac{1}{a(v,u)}C^{(1)}(v)D^{(1)}(u) - \frac{b(v,u)}{a(v,u)}C^{(1)}(u)D^{(1)}(v)\,, \\ C^{(1)}(u)C^{(1)}(v) &= C^{(1)}(v)C^{(1)}(u)\,. \end{aligned} \tag{A.13}$$

Applying the nested transfer matrix (A.8) to the nested Bethe state (A.10), one obtains the eigenvalue of the nested system:

$$\varepsilon^{(1)}(u) = \prod_{i=1}^n a(u,v_i)\prod_{j=1}^m \frac{1}{a(u,v_j^{(1)})} + \prod_{j=1}^m \frac{1}{a(v_j^{(1)},u)}\,, \tag{A.14}$$

where $v_j^{(1)}$ is constrained by the nested Bethe ansatz equations:

$$\prod_{\alpha=1,\neq\beta}^m \frac{v_\beta^{(1)} - v_\alpha^{(1)} + \eta}{v_\beta^{(1)} - v_\alpha^{(1)} - \eta} = \prod_{\gamma=1}^n \frac{v_\beta^{(1)} - v_\gamma - \frac{\eta}{2}}{v_\beta^{(1)} - v_\gamma + \frac{\eta}{2}} \qquad (\beta = 1,2,\ldots,m)\,. \tag{A.15}$$

Then from (2.10) and (A.6) and (A.7), we obtain the eigenvalue $\varepsilon(u)$ of the supersymmetric t-J model: $t(u)\Phi = \varepsilon(u)\Phi$ with

$$\begin{aligned}\varepsilon(u) = &\prod_{i=1}^{N} a(u,z_i) \prod_{j=1}^{n} \frac{1}{a(u,v_j)} \left(\prod_{i=1}^{n} a(u,v_i) \prod_{j=1}^{m} \frac{1}{a(u,v_j^{(1)})} + \prod_{j=1}^{m} \frac{1}{a(v_j^{(1)},u)} \right) - \\ &- \prod_{i=1}^{N} c(u,z_i) \prod_{j=1}^{n} \frac{c(v_j,u)}{a(v_j,u)} . \end{aligned} \tag{A.16}$$

Here the unwanted terms in (A.6) and (A.7) vanish when we take supertrace of the transfer matrix, which yields the Bethe ansatz equations

$$\prod_{i=1}^{N} \frac{v_\beta - z_i - \eta}{v_\beta - z_i} \prod_{\alpha=1,\neq\beta}^{n} \frac{v_\alpha - v_\beta - \eta}{v_\alpha - v_\beta + \eta} \prod_{\gamma=1}^{m} \frac{v_\gamma^{(1)} - v_\beta}{v_\gamma^{(1)} - v_\beta + \eta} = 1 \qquad (\beta = 1, 2, \dots, m) . \tag{A.17}$$

B. The exchange symmetry of the Bethe vector

For the Bethe vector $\Phi_N(v_1, \dots, v_n)$ of the supersymmetric t-J model, we define the exchange operator $\hat{f}_\sigma = \hat{f}_{\sigma_1} \dots \hat{f}_{\sigma_k}$ by

$$\hat{f}_\sigma \Phi_N(v_1, v_2, \dots, v_n) = \Phi_N(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n)}) , \tag{B.1}$$

where $\sigma \in \mathcal{S}_n$ and σ_i are elementary permutations. In [13]–[15], it has been shown that the $gl(m)$ Bethe vector is invariant under the action of the exchange operator $\hat{f}_\sigma$. In this appendix, we examine the exchange symmetry of the $gl(2|1)$ Bethe vector.

We first study the exchange symmetry for the elementary exchange operator $\hat{f}_{\sigma_i}$ which exchanges the parameter v_i and v_{i+1}. Acting $\hat{f}_{\sigma_i}$ on the Bethe vector of $gl(2|1)$ (5.2), we have

$$\begin{aligned}\hat{f}_{\sigma_i} \Phi_N(v_1, v_2, \dots, v_n) &= \Phi_N(v_1, \dots, v_{i+1}, v_i, \dots, v_n) \\ &= \sum_{d_1,\dots,d_n} (\phi_n^{(1),\sigma_i})^{d_1 \dots d_n} C_{d_1}(v_1) \dots C_{d_i}(v_{i+1}) C_{d_{i+1}}(v_i) \dots C_{d_n}(v_n) |0\rangle ,\end{aligned} \tag{B.2}$$

where $(\phi_n^{(1),\sigma_i})^{d_1 \dots d_n}$ is constructed by the nested monodromy matrix

$$T^{(1),\sigma_i}(u) = L_n^{(1)}(u, v_n) \cdots L_{i+1}^{(1)}(u, v_i) L_i^{(1)}(u, v_{i+1}) \cdots L_1^{(1)}(u, v_1). \tag{B.3}$$

The commutation relation between C_i and C_j in (A.4) can be rewritten as

$$C_i(u) C_j(v) = - \sum_{k,l} \frac{\check{r}(u,v)_{ij}^{kl}}{c(u,v)} C_k(v) C_l(u) \tag{B.4}$$

by using the braided r-matrix $\check{r}(u,v) \equiv \mathcal{P} r(u,v)$, where $\mathcal{P}$ permutes the tensor spaces of the 2-dimensional $gl(2)$-module. Then, by (B.4), (B.2) becomes

$$\begin{aligned}\hat{f}_{\sigma_i} \Phi_N(v_1, v_2, \dots, v_n) = -c(v_i, v_{i+1}) &\sum_{d_1,\dots,d_n} (\phi_n^{(1),\sigma_i})^{d_1 \dots d_n} C_{d_1}(v_1) \dots \times \\ &\times (\check{r}(v_{i+1}, v_i))_{d_i d_{i+1}}^{k\ \ l} C_k(v_i) C_l(v_{i+1}) \dots C_{d_n}(v_n) |0\rangle.\end{aligned} \tag{B.5}$$

We now compute the action of $(\check{r}(v_{i+1},v_i))_{d_i d_{i+1}}^{k\ \ l}$ on $(\phi^{(1),\sigma_i})^{d_1\ldots d_n}$. One checks that $\check{r}$-matrix satisfies the YBE

$$\check{r}_{i\,i+1}(v_{i+1},v_i)L_{i+1}^{(1)}(u,v_i)L_i^{(1)}(u,v_{i+1}) = L_{i+1}^{(1)}(u,v_{i+1})L_i^{(1)}(u,v_i)\check{r}_{i\,i+1}(v_{i+1},v_i)\,. \tag{B.6}$$

Therefore, acting $\check{r}$ on $T^{(1),\sigma_i}(u)$, we have

$$\check{r}_{i\,i+1}(v_{i+1},v_i)T^{(1),\sigma_i}(u) = T^{(1)}(u)\check{r}_{i\,i+1}(v_{i+1},v_i)\,. \tag{B.7}$$

Thus, because $\check{r}\,v_2\otimes v_2 = v_2\otimes v_2$, we obtain

$$(\phi_n^{(1)})^{d_1\ldots kl\ldots d_n} = \sum_{d_i d_{i+1}} (\check{r}(v_{i+1},v_i))_{d_i d_{i+1}}^{k\ \ l}(\phi_n^{(1),\sigma_i})^{d_1\ldots d_i d_{i+1}\ldots d_n}\,. \tag{B.8}$$

Changing the indices k,l to d_i,d_{i+1}, respectively, and substituting the above relation into (B.5), we obtain the exchange symmetric relation of the Bethe vector of $gl(2|1)$

$$\hat{f}_{\sigma_i}\Phi_N(v_1,v_2,\ldots,v_n) = -c(v_i,v_{i+1})\Phi_N(v_1,v_2,\ldots,v_n) \tag{B.9}$$

for the elementary permutation operator σ_i.

It follows that under the action of the exchange operator f_σ

$$\hat{f}_\sigma\Phi_N(v_1,v_2,\ldots,v_n) = \mathrm{sign}(\sigma)c^\sigma_{1\ldots n}\Phi_N(v_1,v_2,\ldots,v_n), \tag{B.10}$$

where $\mathrm{sign}(\sigma)=1$ if σ is even and $\mathrm{sign}(\sigma)=-1$ if σ is odd, and $c^\sigma_{1,\ldots,n}$ has the decomposition law

$$c^{\sigma'\sigma}_{1\ldots n} = c^\sigma_{\sigma'(1\ldots n)}c^{\sigma'}_{1\ldots n} \tag{B.11}$$

with $c^{\sigma_i}_{1\ldots n} = c_{i\ \,i+1}\equiv c(v_i,v_{i+1})$ for an elementary permutation σ_i.

C. An example of (5.13)

As an illustration, we give a detailed derivation of (5.13) for the $n=2$ case. In the F-basis, the $gl(2|1)$ Bethe vector is given by

$$\tilde{\Phi}_N(v_1,v_2) = s(c)\sum_{d_1,d_2}(\phi_2^{(1)})^{d_1 d_2}\tilde{C}_{d_1}(v_1)\tilde{C}_{d_2}(v_2)|0\rangle \tag{C.1}$$

with $d_1,d_2=1,2$. The quantum number p_1 may take three values 0,1 or 2. Here we concentrate on the $p_1=1$ case and the $p_1=0,2$ cases can be treated similarly. We have

$$\begin{aligned}\frac{1}{s(c)}\tilde{\Phi}_N^{p_1=1}(v_1,v_2) &= (\phi_2^{(1)})^{12}\tilde{C}_1(v_1)\tilde{C}_2(v_2)|0\rangle + (\phi_2^{(1)})^{21}\tilde{C}_2(v_1)\tilde{C}_1(v_2)|0\rangle\\ &= g(v_1,v_2)(\phi_2^{(1)})^{12}\tilde{C}_2(v_2)\tilde{C}_1(v_1)|0\rangle + \\ &\quad + \left[g'(v_1,v_2)(\phi^{(1)})^{12} + (\phi_2^{(1)})^{21}\right]\tilde{C}_2(v_1)\tilde{C}_1(v_2)|0\rangle\,,\end{aligned} \tag{C.2}$$

where $g(v_1, v_2) = -c(v_2, v_1)/a(v_2, v_1)$, $g'(v_1, v_2) = -b(v_2, v_1)/a(v_2, v_1)$. Acting $\hat{f}_{\sigma_1}$ on (C.2), we have

$$\begin{aligned}\frac{1}{s(c)}\hat{f}_{\sigma_1}\tilde{\Phi}_N^1(v_1, v_2) &= \frac{1}{s(c)}\tilde{\Phi}_N^1(v_2, v_1)\\ &= g(v_2, v_1)(\phi_2^{(1),\sigma_1})^{12}\tilde{C}_2(v_1)\tilde{C}_1(v_2)|0\rangle +\\ &+ \left[g'(v_2, v_1)(\phi_2^{(1),\sigma_1})^{12} + (\phi_2^{(1),\sigma_1})^{21}\right]\tilde{C}_2(v_2)\tilde{C}_1(v_1)|0\rangle\,,\end{aligned} \tag{C.3}$$

Now the exchange symmetry

$$\tilde{\Phi}_N^1(v_2, v_1) = -c(v_1, v_2)\tilde{\Phi}_N^1(v_1, v_2)$$

gives rise to the relation:

$$-c(v_2, v_1)g(v_2, v_1)(\phi_2^{(1),\sigma_1})^{12} = g'(v_1, v_2)(\phi_2^{(1)})^{12} + (\phi_2^{(1)})^{21}\,. \tag{C.4}$$

By means of (C.4), one may recast (C.2) into the form

$$\begin{aligned}\frac{1}{s(c)}\tilde{\Phi}_N^1(v_1, v_2) &= g(v_1, v_2)(\phi_2^{(1)})^{12}\tilde{C}_2(v_2)\tilde{C}_1(v_1)|0\rangle -\\ &- c(v_2, v_1)g(v_2, v_1)(\phi_2^{(1),\sigma_1})^{12}\tilde{C}_2(v_1)\tilde{C}_1(v_2)|0\rangle\,,\end{aligned} \tag{C.5}$$

which coincides with (5.13) for $n = 2$.

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